

Stress Propagation in Two Dimensional Frictional Granular Matter

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The stress profile and reorientation of grains, in response to a point force applied to a preloaded two dimensional granular system, are calculated in the context of a continuum theory that incorporates the texture of the packing. When high friction prevents slip at the inter-grain contacts, an anisotropic packing propagates stress along two peaks which amalgamate into a single peak as the packing is disordered into a less anisotropic structure; this single peak may be wider or narrower than in the isotropic case, depending on the preparation of the packing. At lower frictions, an effective treatment of slipping contacts yields sharpened peaks, and ultimately a singular limit in which stress propagates along straight rays. Recent experiments, as well as aspects of hyperbolic models, are discussed in light of these results.

Descriptions of static granular matter as an isotropic elasto-plastic continuum [1] do not incorporate effects of granularity essential to reproduce the variety of stress profiles identified experimentally [2–9]. By contrast, recent discrete models relying on (probabilistic) rules of force transmission among individual grains [2,10,11] require, for a continuum limit in terms of stresses, an additional constraint relating the components of the stress tensor. This closure relation may be interpreted [11] as a local Janssen approximation [12,1] but is otherwise rather *ad hoc*. Furthermore, it leads to hyperbolic equations for stress propagation [13,14,11], as opposed to the usual elliptic equations of elastic theories.

Reference [15] offers a possible bridging approach in which a continuum theory is derived systematically from grain-grain interactions, taking into account the texture of the packing and the freedom of grains to reorient with respect to their surrounding medium. In the present Letter, we apply this general framework to calculate the propagation of the stress from a point force applied to a two dimensional granular system. After recalling the main elements of the theory and discussing its general properties in two dimensions, we calculate stress profiles in the presence of a friction large enough to prevent slip at the contacts. We obtain a variety of responses that depend on the type of packing and provide us with a natural interpretation of recent experiments. Finally, we discuss the case of easily sheared contacts, which mimic slipping contacts and induce sharper stress peaks. In the limiting situation, the hyperbolic closure relations used in the literature [13,14,11] emerge from our formulation and constrain the applied point force to propagate along straight rays.

As shown in Ref. [15], a strain ε about a preloaded equilibrium state induces reorientation of the grains, expressed by the linearized rotation matrix $\delta_{ij} + \Omega_{ij}$, and generates incremental stresses

$$\sigma_{ij} = (\varepsilon_{ik} - \Omega_{ik}) Q_{kj} + P_{ijkl} \varepsilon_{kl}. \quad (1)$$

The fabric tensors

$$Q_{kj} = \int k_{\perp} D_k D_j \mu d\alpha, \quad (2)$$

$$P_{ijkl} = \int (k_{\parallel} - k_{\perp}) \frac{D_i D_j D_k D_l}{D^2} \mu d\alpha, \quad (3)$$

encode the texture of the granular packing *via* the average number of inter-grain contacts $\mu(\alpha)d\alpha$ within an angle $d\alpha$ about α , the normal and shear moduli $k_{\parallel}(\alpha)$ and $k_{\perp}(\alpha)$ of these contacts, and the average center-to-center distances $D(\alpha)$ traversing them. In two dimensions Ω depends on a single angle ω , defined by $\Omega = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$ and set by the local vanishing of torque to

$$\omega = \frac{(Q_{xx} - Q_{yy})\varepsilon_{xy} + Q_{xy}(\varepsilon_{yy} - \varepsilon_{xx})}{Q_{xx} + Q_{yy}} = \frac{[Q, \varepsilon]_{xy}}{\text{Tr}(Q)}. \quad (4)$$

The stress then can be expressed in terms of the strain alone, for example in the matrix form

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = M \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{pmatrix}, \quad (5)$$

where the (symmetric) compliance matrix M summarizes the textured stiffness of the medium through the first few Fourier moments of the quantities $\mu_Q(\alpha) = k_{\perp}(\alpha) D^2(\alpha) \mu(\alpha)$ and $\mu_P(\alpha) = [k_{\parallel}(\alpha) - k_{\perp}(\alpha)] D^2(\alpha) \mu(\alpha)$ associated with the two fabric tensors. Defining the Fourier moments by

$$\begin{aligned} 2\pi\mu_Q(\alpha) &= 2\kappa_0 + 4\kappa_1 \cos(2\alpha) + 4\kappa_2 \sin(2\alpha) + \dots, \\ 2\pi\mu_P(\alpha) &= 2\bar{\kappa}_0 + 4\bar{\kappa}_1 \cos(2\alpha) + 4\bar{\kappa}_2 \sin(2\alpha) \\ &\quad + 16\bar{\kappa}_3 \cos(4\alpha) + 16\bar{\kappa}_4 \sin(4\alpha) + \dots, \end{aligned} \quad (6)$$

we write M explicitly, as

$$M = \begin{pmatrix} \tilde{\kappa}_0 + \tilde{\kappa}_1 + \tilde{\kappa}_3 & -\tilde{\kappa}_3 & \tilde{\kappa}_2 + \tilde{\kappa}_4 \\ -\tilde{\kappa}_3 & \tilde{\kappa}_0 - \tilde{\kappa}_1 + \tilde{\kappa}_3 & \tilde{\kappa}_2 - \tilde{\kappa}_4 \\ \tilde{\kappa}_2 + \tilde{\kappa}_4 & \tilde{\kappa}_2 - \tilde{\kappa}_4 & -\tilde{\kappa}_3 + \tilde{\kappa}_5 \end{pmatrix}, \quad (7)$$

where $\tilde{\kappa}_0 = \kappa_0 + \bar{\kappa}_0$, $\tilde{\kappa}_1 = \kappa_1 + \bar{\kappa}_1$, $\tilde{\kappa}_2 = (\kappa_2 + \bar{\kappa}_2)/2$, $\tilde{\kappa}_3 = \bar{\kappa}_3 - \bar{\kappa}_0/4 - \kappa_2^2/2\kappa_0$, $\tilde{\kappa}_4 = \bar{\kappa}_4 + \kappa_1\kappa_2/2\kappa_0$, and $\tilde{\kappa}_5 = \kappa_0/2 - (\kappa_1^2 + \kappa_2^2)/2\kappa_0$. The fore-aft symmetry ($\alpha \rightarrow \alpha + \pi$) of the contact distribution forbids dipolar moments, and the anisotropic part of M is composed of the quadrupolar moments κ_1 , $\bar{\kappa}_1$, κ_2 , $\bar{\kappa}_2$ and octupolar moments $\bar{\kappa}_3$, $\bar{\kappa}_4$.

Before discussing stress profiles generated by Eq. (5), we comment on the properties of the compliance. The non-negativity of μ_Q and μ_P restricts the allowed domain of the multipolar components; $\bar{\kappa}_0$, $\bar{\kappa}_1$ and $\bar{\kappa}_3$, for example, are bound by $2(\bar{\kappa}_1/\bar{\kappa}_0)^2 - 1 \leq 4\bar{\kappa}_3/\bar{\kappa}_0 \leq 1$. An intuitive derivation of these (Cauchy-Schwartz) inequalities uses a representation of μ_Q and μ_P as linear combinations of delta functions with non-negative weights. The multipolar components of μ_Q and μ_P then are themselves linear combinations of the multipolar components of the delta functions with the same non-negative weights, hence the former interpolate the latter. For a (fore-aft symmetric) pair of delta functions $\mu_{\alpha_0}(\alpha) = \delta(\alpha - \alpha_0) + \delta(\alpha - \alpha_0 - \pi)$, $\bar{\kappa}_0(\mu_{\alpha_0}) = 1$, $\bar{\kappa}_1(\mu_{\alpha_0}) = \cos(2\alpha_0)$, $\bar{\kappa}_3(\mu_{\alpha_0}) = \cos(4\alpha_0)/4$, and the bound $2(\bar{\kappa}_1/\bar{\kappa}_0)^2 - 1 \leq 4\bar{\kappa}_3/\bar{\kappa}_0$ is achieved. The choice $\mu(\alpha) = \mu_{\alpha_0}(\alpha)$ corresponds to the rather pathological limit of a granular medium made up of an array of independent, parallel columns at an angle α_0 , and lies on the boundary of the allowed domain. This domain is included entirely in the domain of stability of M , in which stress resists any deformation (with a positive energy cost). As long as friction is high enough to prevent slip at the contacts, the boundaries of the domain of stability and the allowed domain meet only at the columnar systems just mentioned. Finally, the form of M in Eq. (7) and the independence of $\tilde{\kappa}_0, \dots, \tilde{\kappa}_5$ ensure that, within the allowed domain, the granular material can take on the most general form of anisotropic elasticity.

The equilibrium conditions $\partial_j \sigma_{ij} = f_i \delta(x) \delta(y)$, where f_i is a point force applied at the origin, are supplemented by the closure equation $\partial_{xx} \varepsilon_{yy} + \partial_{yy} \varepsilon_{xx} = 2\partial_{xy} \varepsilon_{xy}$. This relation specifies the constraint on a strain tensor that derives from an underlying two-component displacement vector field, and is re-expressed as

$$(\partial_{yy}, \partial_{xx}, -\partial_{xy}) M^{-1} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = 0 \quad (8)$$

to close the system of equations governing stresses. A classic and elegant solution makes use of the Airy function ϕ , defined through its derivatives by $\delta_{ij} \partial_{\kappa\kappa} \phi - \partial_i \partial_j \phi = \sigma_{ij}$ so as to satisfy the equilibrium conditions automatically. Thus one needs to solve only Eq. (8) for

ϕ , respecting the boundary conditions at the origin and at infinity. Taking advantage of the properties of analytic functions on the plane, one can translate this fourth order differential equation for ϕ into a fourth order algebraic equation [16]. Its roots determine completely the angular modulation that factors the usual $1/r$ dependence of two dimensional elastic stresses. In an orthotropic material (with two perpendicular axes of symmetry), the quartic polynomial in question reduces to a bi-quadratic one [17], and henceforth we restrict ourselves to this case for the sake of simplicity. Specifically, we choose the x - and y -axes as axes of symmetry, so that $\kappa_2 = \bar{\kappa}_2 = \bar{\kappa}_4 = 0$.

Following the program just outlined, we calculate the radial stress σ_{rr} and the reorientation angle ω analytically and plot their angular dependence on Fig. 1 [18]. While observed modulated (*e.g.*, double-peaked) responses motivate hyperbolic models [11,13,14], it was emphasized recently [20] that they arise also in anisotropic elastic systems [17,16,21]. Accordingly, for a weakly disordered square lattice oriented at 45° from the x - and y -axes ($\bar{\kappa}_3 < 0$), we find that the point force propagates through the material as a shallow stress double-peak (middle bold curve in Fig. 1(a)). As the packing is further disordered, the two peaks merge into a single peak which ultimately coincides with the isotropic response (thin solid curve). Alternatively, if contacts close to the vertical are slightly favored ($\kappa_1, \bar{\kappa}_1 < 0$) *e.g.*, by gravity, the two peaks amalgamate into a single wide peak (top bold curve). If, however, the preparation of the system favors contacts close to the horizontal ($\kappa_1, \bar{\kappa}_1 > 0$), the two peaks become more pronounced (bottom bold curve) and the stress propagates mostly along two directions. (We note that there is no trivial relation between the latter and possible preferred directions in the packing.)

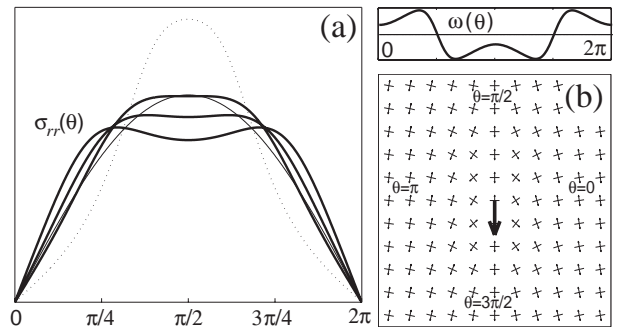


FIG. 1. Radial stress and reorientation angle as a function of the polar angle, for different packings of incompressible grains ($\kappa_1 = 2\bar{\kappa}_1$ [19,15]). (a) Bold curves: $\bar{\kappa}_3 = -1/5$; $\bar{\kappa}_1 = -1/5$ (top), $\bar{\kappa}_1 = 0$ (middle), $\bar{\kappa}_1 = 1/5$ (bottom). Dotted curve: $\bar{\kappa}_1 = -7/10$, $\bar{\kappa}_3 = 0$. The isotropic result (thin solid line) is added for comparison. (b) $\bar{\kappa}_1 = 1/5$, $\bar{\kappa}_3 = -1/5$. The crosses illustrate the angular dependence of the reorientation and its $1/r$ decay.

The plots are symmetric because they correspond to

packings statistically symmetric about the vertical (direction of the force). If the material is rotated with respect to the force, one of the two peaks dominates the other; more generally, if the force is not applied along an axis of symmetry (as generically in a non-orthotropic medium), we expect a biased response reminiscent of arching phenomena [22]. This is certainly the case in the left (or right) half of a pile constructed with a point source, where the packing is not symmetric about the vertical [8,15]. When $\kappa_1 \neq 0$, the stress profile is due in part to the reorientation of the grains. Figure 1(b) exhibits the angular dependence of the reorientation angle ω corresponding to the pronounced double-peak in Fig. 1(a); the crosses illustrate the form of the reorientation field and its decay away from the origin. While the reorientation appears to relieve the shear stress along the vertical as expected intuitively, it shows a complicated structure and in particular extrema along non-trivial directions.

In a remarkable measurement [7] of the response of a vertical two dimensional granular system (preloaded by its own weight) to a vertical point force, the stress propagates along two peaks, in an ordered (anisotropic) piling, which merge into a single peak as the piling is disordered, in conformity with our picture. Furthermore, the width of the single peak increases linearly with depth, in accordance with an elastic theory. (The dependence of the widths on depth in the case of a double-peak is not given.) Another experiment [8] correlates the macroscopic response to the microscopic structure by recording both the stress profile and the angular distribution of contacts $\mu(\alpha)$. As expected [15], $\mu(\alpha)$ shows a higher degree of anisotropy in a pile grown with a point source than in a pile grown with an extended source, and different stress profiles are measured: a double-peak for a point source, a single peak for an extended source. In similar experiments on three dimensional piles [3,5] the observed stress peak, while scaling linearly in depth, is narrower than expected from isotropic elasticity in an infinite half space [5]. The authors interpret this accentuated response as possibly arising from the experimental boundary conditions [5]—a conjecture left unconfirmed by a more detailed study [6]. Our approach affords a complementary explanation: no matter how one disorders the pile during or after its growth, gravity sets a preferred direction; one then expects to have more near-vertical contacts than near-horizontal ones, and a resulting narrower response (dotted curve in Fig. 1(a)). This interpretation is supported by measurements [6] showing that a dense packing yields a wider peak than a loose packing. To go from dense to loose, one needs to “open holes” while maintaining stability; thus these holes may decrease the number of near-horizontal contacts relative to the number of near-vertical ones, but not the other way around.

To complete our picture of stress propagation, we discuss the degenerate solutions that arise when slip

occurs. As long as a large friction prevents slip, the ratio of shear to normal moduli of the contacts is fixed (for purely compressively preloaded contacts, $k_{\perp}/k_{\parallel} = (2 - 2\nu)/(2 - \nu)$, where ν is the Poisson ratio of the constitutive material of the grains [19]). At smaller frictions, grains may slip along their contacts, effectively lowering the value of k_{\perp} [23]. This renormalization of the shear modulus yields narrower stress peaks, and in the limit $k_{\perp}/k_{\parallel} \rightarrow 0$ singular responses arise in ordered packings. For a simple two dimensional crystal (Bravais lattice) $\mu_Q(\alpha), \mu_P(\alpha) \propto \sum_{i=1}^n [\delta(\alpha - \alpha_i) + \delta(\alpha - \alpha_i - \pi)]$, where steric constraints impose $n \leq 3$ [24]. The case $n = 1$ corresponds to the pathological columnar systems mentioned above, while $n = 3$ yields an isotropic elastic theory since the contact angles differ by multiples of $\pi/3$. In the case $n = 2$, the stress peaks become infinitely narrow and coincide with the lattice directions α_1 and α_2 . Similar ray-like responses have been observed recently in three dimensional crystalline packings [9]. In two dimensions, the situation is easiest to visualize in a square lattice at 45° (Fig. 2); a point force applied vertically is transmitted along two rows of grains without disturbing any other grains. We note that, in general, in the limit $k_{\perp}/k_{\parallel} \rightarrow 0$ only the fabric tensor P survives in Eq. (1) and the system becomes equivalent to a collection of fibers highly resistant to compression and extension but easy to shear, as in much studied fiber-reinforced materials [25]. Ordering corresponds to aligning the fibers along a set of given directions; in the case of only two directions there is at least one soft mode, associated with a singular response. In particular, the conformation of Fig. 2 is equivalent to two sets of fibers at right angle, equivalent in turn to an incompressible elastic medium with fibers along a single direction.

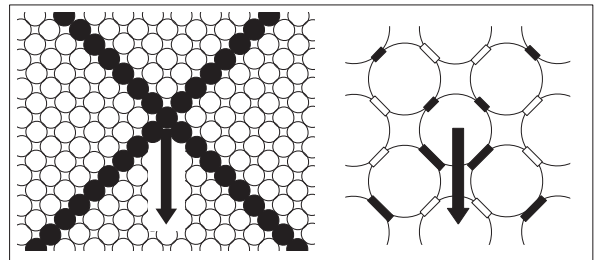


FIG. 2. Illustration of a singular response in an ordered packing. Left: the force is transmitted through the two rows of black grains. Right: detail; over- and under-compressed contacts are in black, unperturbed ones in white.

Singular solutions of a hyperbolic character occur only at the boundary of the domain of stability of the elastic theory. In the limit $k_{\perp}/k_{\parallel} \rightarrow 0$, this boundary and that of the allowed domain meet not only at the $n = 1$ columnar systems but also at the $n = 2$ crystalline packings. For a packing symmetric with respect to the x - and y -axes ($\alpha_1 = -\alpha_2 \equiv \alpha_0$), the closure Eq. (8) reduces

to

$$(\partial_{xx} - c_0^{-2} \partial_{yy})(\sigma_{xx} - c_0^2 \sigma_{yy}) = 0, \quad (9)$$

where $c_0^{-2} = (\bar{\kappa}_0 - \bar{\kappa}_1)/(\bar{\kappa}_0/4 - \bar{\kappa}_3) - 1 = \tan^2 \alpha_0 \geq 0$. If the system is rotated by an angle $\arctan(t)$, the closure equation involves the more general linear combination $(1 - c_0^2 t^2) \sigma_{xx} - (c_0^2 - t^2) \sigma_{yy} - 2(1 + c_0^2) t \sigma_{xy}$. Since the limit $k_\perp/k_\parallel \rightarrow 0$ may be viewed also as that of infinitely hard grains (with a thin incompressible elastic coating), it is no surprise that we retrieve the form of the closure equations derived [26,27] for the marginal (isostatic) state of a granular packing in which the response may be calculated without knowledge of deformations. In integral form, our closure equation with $c_0^2 = 1$ is identical to the one given in Ref. [26], while for general c_0 and t it coincides with that of Ref. [27] and the phenomenological “oriented stress linearity model” of Ref. [14]. In Ref. [11], an integral form of Eq. (9) is suggested by a discrete probabilistic model and the packing disorder is emulated by adding to c_0^2 a small increment that fluctuates randomly in space. This uncertainty in the local propagation directions results in a diffusive broadening of the stress rays, proportional to the square root of the depth. Within our approach, by contrast, as soon as disorder in the contact orientation is introduced, the response of a homogeneous system is governed by elastic (elliptic) equations on large scales, yielding a linear broadening of the stress peaks. These remarks help conciliate the square-root broadening observed [4] in a pile less than ten grains in height with the linear broadening [5–7] on larger scales. The situation might be different, however, for heterogeneous media; in a polycrystalline packing, for example, one might expect a diffusive behavior even on macroscopic scales.

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- [1] For reviews see, *e.g.*, R.M. Nedderman, *Statics and Kinematics of Granular Materials* (Cambridge Univ. Press, Cambridge, 1992) and S.B. Savage, in *Physics of Dry Granular Media*, H.J. Herrmann, J.-P. Hovi, and S. Luding, Eds. (Kluwer, Dordrecht, 1998).
 - [2] C.-h. Liu, S.R. Nagel, D.A. Scheeter, S.N. Coppersmith, S. Majumdar, O. Narayan, and T.A. Witten, *Science* **269**, 513 (1995).
 - [3] L. Vanel, D.W. Howell, D. Clark, R.P. Behringer, and E. Clément, *Phys. Rev. E* **60**, R5040 (1999).
 - [4] M. Da Silva and J. Rajchenbach, *Nature* **406**, 708 (2000).
 - [5] G. Reydellet and E. Clément, *Phys. Rev. Lett.* **86**, 3308 (2001).
 - [6] D. Serero, G. Reydellet, P. Claudin, and E. Clément, *Eur. Phys. J. E* **6**, 169 (2001).

- [7] J. Geng *et al.*, *Phys. Rev. Lett.* **87**, 035506 (2001).
- [8] J. Geng, E. Longhi, R.P. Behringer, and D.W. Howell, *Phys. Rev. E* **64**, 060301 (2001).
- [9] N.W. Mueggenburg, H.M. Jeager, and S.R. Nagel, preprint (cond-mat/0204533).
- [10] S.N. Coppersmith, C.-h. Liu, S. Majumdar, O. Narayan, and T.A. Witten, *Phys. Rev. E* **53**, 4673 (1996).
- [11] P. Claudin, J.-P. Bouchaud, M.E. Cates, and J.P. Wittmer, *Phys. Rev. E* **57**, 4441 (1998), and references therein.
- [12] H.A. Janssen, *Z. Vert. Dt. Ing.* **39**, 1045 (1895).
- [13] J.P. Wittmer, M.E. Cates, P. Claudin, and J.-P. Bouchaud, *Nature* **382**, 336 (1996).
- [14] J.P. Wittmer, M.E. Cates, and P. Claudin, *J. Phys. I* **7**, 39 (1997).
- [15] C. Gay and R. da Silveira, preprint (cond-mat/0208155).
- [16] A.E. Green and W. Zerna, *Theoretical Elasticity* (Oxford Univ. Press, London, 1954).
- [17] A.E. Green and G.I. Taylor, *Proc. Roy. Soc. Lond. A* **173**, 162 (1939).
- [18] The results shown in Fig. 1 are derived at strong anisotropy, for which the approach of Ref. [15] is best suited. At weak anisotropy, $\bar{\kappa}_1$ and $\bar{\kappa}_3$ correct the isotropic response according to

$$\frac{2\pi r}{17/11} \sigma_{rr} = f_x \cos \theta + f_y \sin \theta + \frac{48}{187} \left\{ 3\bar{\kappa}_1 \left[f_x \cos \theta \left(-\frac{7}{24} + \cos 2\theta \right) + f_y \sin \theta \left(\frac{7}{24} + \cos 2\theta \right) \right] + \bar{\kappa}_3 \left[f_x \cos \theta \left(-\frac{7}{15} + \cos 2\theta + \frac{3}{5} \cos 4\theta \right) + f_y \sin \theta \left(-\frac{7}{15} - \cos 2\theta + \frac{3}{5} \cos 4\theta \right) \right] \right\}, \quad (10)$$
 where we neglected higher order terms and assumed incompressible grains ($k_\parallel/k_\perp = 2/3$, $\kappa_1 = 2\bar{\kappa}_1$ [19,15]).
- [19] K.L. Johnson, *Contact Mechanics* (Cambridge Univ. Press, Cambridge, 1985).
- [20] C. Goldenberg and I. Goldhirsch, preprint (cond-mat/0108297).
- [21] In a very recent preprint, M. Otto, J.-P. Bouchaud, P. Claudin, and J.E.S. Socolar further explore the stress profiles that arise in generic anisotropic elastic theories, and contrast them with stress profiles in force chain systems.
- [22] C.F. Jenkin, *Proc. Roy. Soc. Lond. A* **131**, 53 (1931); D.H. Trollope, in *Rock Mechanics in Engineering Practice*, K.G. Staggs and O.C. Zienkiewicz Eds. (John Wiley, New York, 1968); S.F. Edwards and R.B. Oakeshott, *Physica D* **38**, 88 (1989).
- [23] Low effective values of k_\perp were needed also to conciliate experimental results with an effective medium theory which otherwise largely overestimates the macroscopic shear modulus; see H.A. Makse, N. Glang, D.L. Johnson, and L. Schwartz, *Phys. Rev. Lett.* **83**, 5070 (1999).
- [24] Subtleties associated with more complicated periodic packings are discussed in S. Torquato and F.H. Stillinger, *J. Phys. Chem. B* **105**, 11849 (2001).
- [25] A.C. Pipkin, *Adv. Appl. Mech.* **19**, 1 (1979).
- [26] S.F. Edwards and D.V. Grinev, *Phys. Rev. Lett.* **82**, 5397 (1999).
- [27] R.C. Ball and R. Blumenfeld, *Phys. Rev. Lett.* **88**, 115505 (2002).